LOS ALAMOS SCIENTIFIC LABORATORY

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UNIVERSITY OF CALIFORNIA

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U. S. ATOMIC ENERGY COMMISSION

LA - 1061

Series A

This is copy 14 of 28 copies

January 24, 1950

This document contains 10 pages

NEUTRON DIFFUSION THEORY THE TRANSPORT APPROXIMATION

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General Physics

NEUTRON DIFFUSION THEORY*

THE TRANSPORT APPROXIMATION

One-velocity neutron diffusion problems involving anisotropic scattering are considerably simplified if one applies the so-called Transport Approximation. The latter consists in replacing, in the formulae for isotropic scattering, the scattering cross section σ_s wherever it occurs by the corresponding transport cross section $\bar{\sigma}_s$ to be defined below.

The accuracy of the Transport Approximation has not been thoroughly investigated. In this Report it will be checked against exact theory in the simple problem of calculating the critical radius of a bare sphere in the case of linear anisotropy. The comparison of approximate and exact theory in this special case will give an indication of the error to be expected in similar problems, but good agreement in this case does not of course justify the Transport Approximation in general.

Notation

We consider the processes of scattering, absorption, and fission and denote the corresponding cross sections by σ_s, σ_a, and σ_f, letting $\sigma_t = \sigma_s + \sigma_a + \sigma_f$, ν = the number of neutrons emerging per fission and N the number of nuclei per cm^3. We assume that the fission neutrons are ejected isotropically, and that the scattered neutrons emerge according to the scattering law:

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* This Report is based on a set of notes by G. Placzek.

(1) $$F(\mu) = \frac{1}{2}\left[1 + 3b_1 P_1(\mu) + 5b_2 P_2(\mu) + \dots\dots\right]$$

where $\mu = \cos\theta$, and θ the angle of deflection. In this Report, we restrict the discussion to linear scattering, in which case (1) reduces to:

(2) $$F(\mu) = \frac{1}{2}(1 + 3b_1\mu)$$

where $|b_1| \leq \frac{1}{3}$. We shall refer to b_1 as the "transport" and define $\bar{\sigma}_s$ as follows:

(3) $$\bar{\sigma}_s = \sigma_s(1 - b_1).$$

For the isotropic case we have $b_1 = 0$ and $\bar{\sigma}_s = \sigma_s$. From (2) we also have:

(4) $$\int_{-1}^{1} F(\mu)\,d\mu = 1, \quad \int_{-1}^{1} F(\mu)\,\mu\,d\mu = b_1.$$

Exact Theory

In the exact theory, we work with the following parameters:

(5) $$\begin{cases} \sigma = \sigma_t N \\ c \equiv 1 + f = \dfrac{\sigma_s + \nu\sigma_f}{\sigma_t} \\ b \equiv \dfrac{\alpha}{3c} = \dfrac{1}{c}\cdot\dfrac{\sigma_s}{\sigma_t}\cdot b_1 \end{cases}$$

where b is the effective transport obtained by combining isotropic fission with anisotropic scattering. In addition, the parameter k, a function of c and b, is obtained from the equation:

(6) $$c\,\frac{\arctan k}{k} - \frac{(c-1)\alpha}{k^2}\left(1 - \frac{\arctan k}{k}\right) = 1.$$

Three general methods are available for the calculation of the critical radius σa of a bare sphere. The accuracy of these methods has been investigated in a number of LA Reports. The error in the <u>Endpoint Method</u>[1] increases from 0 to about $0.2\ \%$ as c goes from 1 to 3, and for $c=\infty$ the error is 2.1 %. The error in the <u>Serber-Wilson Method</u>[2] increases rapidly from 0 to about 4 % as c goes from 1 to 1.2. The error then stays nearly constant at about 4 % in the c-interval $(1.2, \infty)$. The error in the <u>Iterative Method</u>[3] with a quadratic trial function decreases from about 0.2% to 0% as c goes from 1.2 to ∞. For c-values very close to 1, the method breaks down.

<u>The Endpoint Method</u>

Using this method we have:

$$\sigma a = \frac{\pi}{k} - z_0 \, . \tag{7}$$

The quantity $k = k(c,b)$ is obtained from equation (6), and $z_0 = z_0(c,b)$ for $b=0$ from <u>LAMS-806</u>:

$$c z_0(c,0) \equiv (1+f_1) z_0(1+f_1, 0) = 1 - (1+f_1)\Delta X(f_1, f_2), \text{ with } f_2 = -1. \tag{8}$$

where the notation of LAMS-806 is used on the right hand side. For $b \neq 0$, the formula for z_0 is given in <u>B.Davison: Memorandum IV</u>,[4] Formula (3.23), but computation using this was not attempted because of the considerable labor involved.

<u>The Serber-Wilson Method</u>

Using this method, σa is obtained as the solution of the equation:

$$\operatorname{Im} E_1(\sigma a(1+ki)) = - e^{-\sigma a} \frac{\alpha \frac{f}{k}}{(1+f) + \alpha \frac{f}{k^2}} \frac{\sin k\sigma a}{k\sigma a} \tag{9}$$

1) LA-258 2) LA-234 3) LA-990 4) Montreal, June 11, 1946.

where the function $\mathrm{Im}\,E_1(x+iy)$ is tabulated in AM-509.

This method has not been used since it is not accurate enough for the purpose of this Report.

The Iterative Method

This method gives σa as the solution of the following matrix eqation, where x denotes σa :

$$(10)\qquad \begin{vmatrix} \frac{1}{3}\left[\frac{1}{c}-\left(P_{00}(x)+\frac{\alpha}{3}\frac{c-1}{c}Q_{00}(x)\right)\right], & \frac{1}{5}\left[\frac{1}{c}-\left(P_{02}(x)+\frac{\alpha}{3}\frac{c-1}{c}Q_{02}(x)\right)\right] \\ \frac{1}{5}\left[\frac{1}{c}-\left(P_{20}(x)+\frac{\alpha}{3}\frac{c-1}{c}Q_{20}(x)\right)\right], & \frac{1}{7}\left[\frac{1}{c}-\left(P_{22}(x)+\frac{\alpha}{3}\frac{c-1}{c}Q_{22}(x)\right)\right] \end{vmatrix} = 0$$

where the functions P_{ij} and Q_{ij} are defined and tabulated in LA-990.

Transport Approximation.

In the Transport Approximation, we work with the following parameters:

$$(11)\qquad \begin{cases} \bar{\sigma} = \sigma_t\left(1-\frac{\alpha}{3}\right)N \\ \bar{c} \equiv 1+\bar{f} = \dfrac{c-\frac{\alpha}{3}}{1-\frac{\alpha}{3}} = 1+\dfrac{c-1}{1-\frac{\alpha}{3}} \\ \bar{b} = 0 \end{cases}$$

where α, $|\alpha| \leq 1$, is defined in (5).

The Endpoint Method

$$(12)\qquad \sigma a = \frac{1}{1-\frac{\alpha}{3}}\left[\frac{\pi}{k(\bar{c})} - z_0(\bar{c})\right].$$

The Serber-Wilson Method

$$(13)\qquad \mathrm{Im}\,E_1\left[\left(1-\frac{\alpha}{3}\right)\sigma a(1+ki)\right] = 0\,,\quad k = k(\bar{c}).$$

The Iterative Method

$$(14)\qquad \begin{vmatrix} \frac{1}{3}\left[\frac{1}{c} - P_{00}(\sigma a)\right], & \frac{1}{5}\left[\frac{1}{c} - P_{02}(\sigma a)\right] \\ \frac{1}{5}\left[\frac{1}{c} - P_{20}(\sigma a)\right], & \frac{1}{7}\left[\frac{1}{c} - P_{22}(\sigma a)\right] \end{vmatrix} = 0 .$$

Expansions for c Close to Unity.

Exact Theory:

$$(15)\qquad \frac{\sigma a_{\alpha}}{\sigma a_{\alpha=0}} = \frac{1}{\sqrt{1-b}}\left\{1 - .3917\left(\frac{1}{\sqrt{1-b}} - 1\right)\sqrt{c-1} + \left[\frac{b}{2(1-b)} - .1534\left(\frac{1}{\sqrt{1-b}} - 1\right)\right](c-1) + \cdots\right\}.$$

Transport Approximation:

$$(16)\qquad \frac{\sigma a_{\alpha}}{\sigma a_{\alpha=0}} = \frac{1}{\sqrt{1-b}}\left\{1 - .3917\left(\frac{1}{\sqrt{1-b}} - 1\right)\sqrt{c-1} + \left[\frac{b}{10(1-b)} - .1534\left(\frac{1}{\sqrt{1-b}} - 1\right)\right](c-1) + \cdots\right\}.$$

Results

The results of the computations are presented in a graph at the end of this Report and in the following tables:

Table Ia	Transport Approximation, σa by the Endpoint Method.
Table Ib	Transport Approximation, σa by the Iterative Method.
Table IIa	Transport Approx., $(1+f)\left[\frac{\sigma a_{\alpha}}{\sigma a_{\alpha=0}} - 1\right]$ by the Endpoint Method.
Table IIb	Transport Approx., $(1+f)\left[\frac{\sigma a_{\alpha}}{\sigma a_{\alpha=0}} - 1\right]$ by the Iterative Method.
Table III	Exact Theory, σa and $(1+f)\left[\frac{\sigma a_{\alpha}}{\sigma a_{\alpha=0}} - 1\right]$ by the Iterative Method.

Table Ia

Transport Approximation, σa by the Endpoint Method

f \ α	-1.0	-0.5	0.0	0.5	1.0
.00	∞	∞	∞	∞	∞
.05	6.408	6.800	7.277	7.875	8.657
.1	4.329	4.576	4.873	5.240	5.709
.2	2.855	3.000	3.172	3.379	3.637
.4	1.8186	1.8961	1.9853	2.0896	2.2139
.6	1.3686	1.4190	1.4759	1.5410	1.6165
.8	1.1069	1.1429	1.1830	1.2280	1.2790
1.0	0.9331	0.9603	0.9901	1.0232	1.0602
1.2	0.8082	0.8295	0.8527	0.8781	0.9061
1.4	.7137	.7309	.7494	.7695	.7914
1.6	.6395	.6536	.6688	.6851	.7027
1.8	.5796	.5914	.6041	.6175	.6320
2.0	.5301	.5402	.5509	.5622	.5742
∞	.0000	.0000	.0000	.0000	.0000

Table Ib

Transport Approximation, σa by the Iterative Method

f \ α	-1.0	-0.5	0.0	0.5	1.0
.00	—	—	—	—	—
.05	—	—	—	—	—
.1	4.353	4.600	4.895	5.258	5.725
.2	2.862	3.007	3.178	3.385	3.642
.4	1.8209	1.8981	1.9871	2.0910	2.2154
.6	1.3696	1.4200	1.4769	1.5420	1.6176
.8	1.1077	1.1436	1.1838	1.2290	1.2803
1.0	0.9336	0.9608	0.9909	1.0242	1.0617
1.2	0.8088	0.8302	0.8535	0.8792	0.9078
1.4	.7143	.7316	.7503	.7707	.7934
1.6	.6401	.6544	.6697	.6864	.7047
1.8	.5802	.5923	.6050	.6190	.6340
2.0	.5308	.5412	.5520	.5638	.5764
∞	.0000	.0000	.0000	.0000	.0000

Table IIa

Transport Approximation, $(4/f)\left[\frac{\delta a_\alpha}{\delta a_{\alpha=0}} - 1\right]$ by the Endpoint Method

f \ α	-1.0	-0.5	0.5	1.0
.00	-.1340	-.0742	0.0954	0.2247
.05	-.1254	-.0688	.0863	.1991
.1	-.1228	-.0670	.0828	.1887
.2	-.1199	-.0651	.0783	.1759
.4	-.1176	-.0629	.0736	.1612
.6	-.1163	-.0617	.0706	.1524
.8	-.1158	-.0610	.0685	.1461
1.0	-.1151	-.0602	.0669	.1416
1.2	-.1148	-.0599	.0655	.1378
1.4	-.1143	-.0592	.0644	.1345
1.6	-.1139	-.0591	.0634	.1318
1.8	-.1136	-.0589	.0621	.1293
2.0	-.1133	-.0583	.0615	.1269
∞	-.1172	-.0586	.0586	.1172

Table IIb

Transport Approximation, $(4/f)\left[\frac{\delta a_\alpha}{\delta a_{\alpha=0}} - 1\right]$ by the Iterative Method

f \ α	-1.0	-0.5	0.5	1.0
.00	–	–	–	–
.05	–	–	–	–
.1	-.1218	-.0663	0 .0816	0 .1865
.2	-.1193	-.0646	.0782	.1752
.4	-.1171	-.0627	.0732	.1608
.6	-.1162	-.0616	.0705	.1524
.8	-.1157	-.0611	.0687	.1467
1.0	-.1157	-.0608	.0672	.1429
1.2	-.1152	-.0601	.0662	.1400
1.4	-.1152	-.0598	.0653	.1379
1.6	-.1149	-.0594	.0648	.1359
1.8	-.1148	-.0588	.0648	.1342
2.0	-.1152	-.0587	.0641	.1326
∞	-.1165	-.0582	.0582	.1165

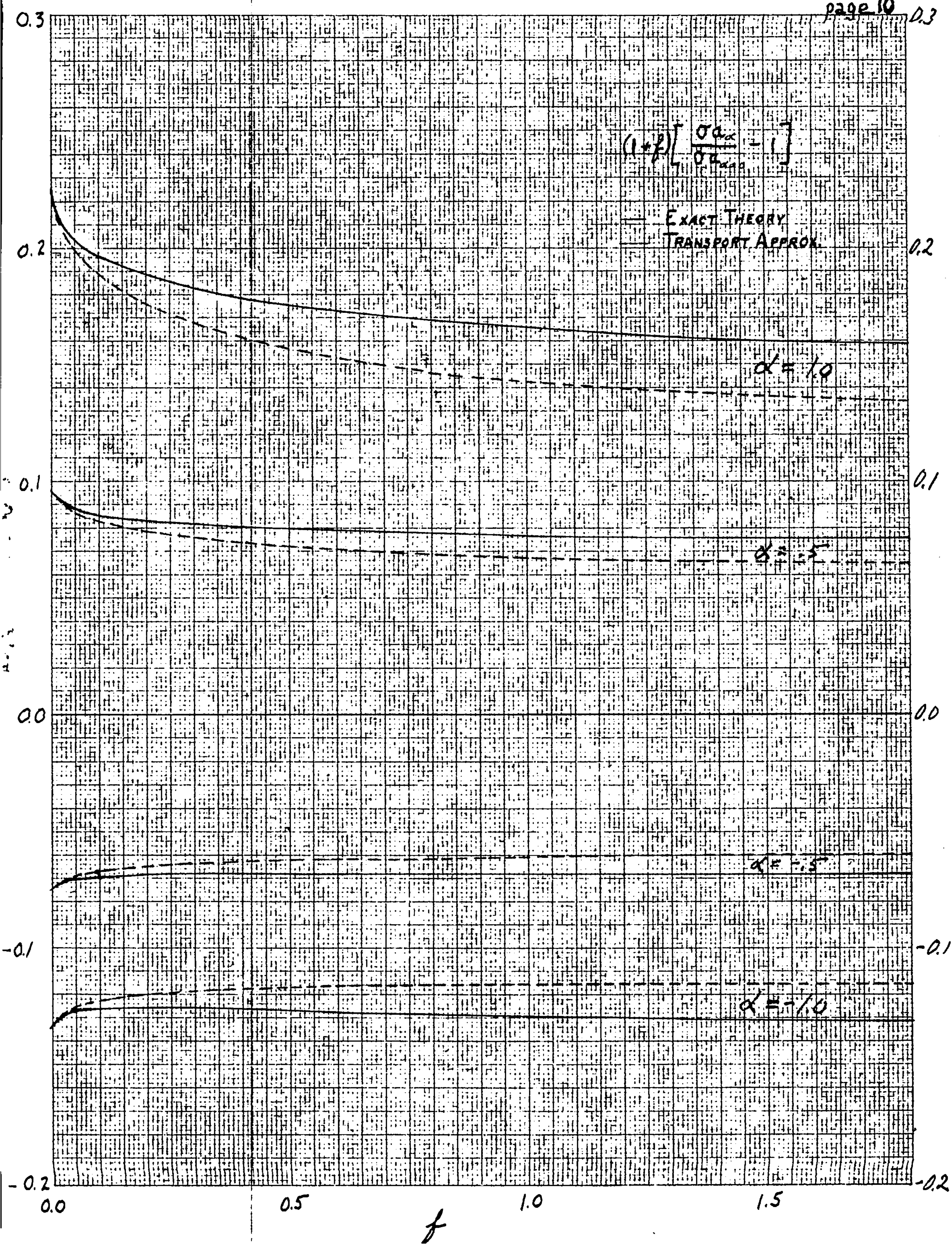
Table III

Exact Theory, σa by the Iterative Method

f \ α	-1.0	-0.5	0.0	0.5	1.0
.00	—	—	—	—	—
.05	—	—	—	—	—
.1	4.333	4.585	4.895	5.276	5.770
.2	2.846	2.997	3.178	3.396	3.678
.4	1.8081	1.8907	1.9871	2.1015	2.2408
.6	1.3593	1.4140	1.4769	1.5498	1.6364
.8	1.0993	1.1388	1.1838	1.2350	1.2946
1.0	0.9269	0.9570	0.9909	1.0290	1.0728
1.2	.8031	.8270	.8535	0.8831	0.9167
1.4	.7094	.7289	.7503	.7740	.8005
1.6	.6359	.6521	.6697	.6892	.7108
1.8	.5766	.5902	.6050	.6212	.6393
2.0	.5276	.5394	.5520	.5656	.5808
∞	.0000	.0000	.0000	.0000	.0000

Exact Theory, $(1+f)\left[\frac{\sigma a_\alpha}{\sigma a_{\alpha=0}} - 1\right]$ by the Iterative Method

f \ α	-1.0	-0.5	0.5	1.0
.00	—	—	—	—
.05	—	—	—	—
.1	-.1263	-.0697	0.0856	0.1966
.2	-.1254	-.0683	.0823	.1888
.4	-.1261	-.0679	.0806	.1787
.6	-.1274	-.0681	.0790	.1728
.8	-.1285	-.0684	.0779	.1685
1.0	-.1292	-.0684	.0769	.1653
1.2	-.1299	-.0683	.0763	.1629
1.4	-.1308	-.0685	.0758	.1606
1.6	-.1312	-.0683	.0757	.1596
1.8	-.1314	-.0685	.0750	.1587
2.0	-.1326	-.0685	.0739	.1565
∞	-.1410	-.0705	.0705	.1410



$(1+f)\left[\frac{\sigma a_\alpha}{\sigma a_{\alpha=0}} - 1\right]$
— EXACT THEORY
— TRANSPORT APPROX.
$\alpha = 1.0$
$\alpha = .5$
$\alpha = -.5$
$\alpha = -1.0$
0.3
0.2
0.1
0.0
-0.1
-0.2
0.0
0.5
1.0
1.5
f